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Conservation Form of the Equations of Fluid Dynamics in General Nonsteady Coordinates

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Introduction

MANY of the differential equations that arise in the field of fluid dynamics may be stated in conservation-law form. A well-known example is, of course, the Navier-Stokes equations.

Recent interest in the generation of general body-oriented curvilinear coordinate systems for solving these equations has given rise to many forms of presentations of the conservation form of equations in curvilinear coordinates. Generally speaking, the conservation-law form of the equations seems definitely preferable, particularly when shocks or other discontinuities form part of the admissible solutions. In the numerical sense, it is also essential to formulate discretization schemes for the governing equations that satisfy conservation requirements.

Several previous works on the subject of deriving the conservation-law form of the Navier-Stokes equations in general nonsteady coordinate systems have been reported in the literature.[1-5] The purpose of this Note is to illustrate a mathematical methodology with which such forms of the equations may be derived in an easier and more general fashion.

For numerical purposes, the scalar form of these equations is eventually presented in Cartesian components. Satisfactory numerical results using the conservation-law form of the equations are impossible to obtain, unless the numerical scheme employed also respects geometric conservation in a discrete sense.

Conservation Form of Equations in Curvilinear Coordinates

The general conservation law for classical fields is stated in integral form[5] as

$$\int_V \frac{\partial A}{\partial t} \mathrm{d}V + \int_S f \cdot n \mathrm{d}S = \int_V C \mathrm{d}V \tag{1}$$

or, in differential form, as

$$\frac{\partial A}{\partial t} + \mathrm{div} f = C \tag{2}$$

where n is the unit outward normal to the surface S which encloses the material volume $V(t)$. The quantities A, f, and C are tensors such that f is of one order higher than A or C, where C is the source term.

As stated in Ref. 5, if Eq. (2) is written (in curvilinear coordinates) in the strong conservation-law form, i.e., in a form in which undifferentiated terms do not appear, then the conservation law will be preserved.

Suppose that physical space $L^3(E_1^c, E_2^c, E_3^c)$§ is spanned by the base vectors E_i^c $(i=1,2,3)$. L^3 may be extended to the four-dimensional linear space $L^4(E_1^c, E_2^c, E_3^c, E_4^c)$, where the vector E_4^c (denoting time t) is defined as satisfying $E_i^c \cdot E_4^c = \delta_{i4}$, δ being the usual Kronecker delta. We now set

$$B = f + AE_4^c = B_c^\alpha E_\alpha^c \tag{3}$$

$$\nabla = E_\alpha^c \frac{\partial}{\partial X^\alpha} \tag{4}$$

Then Eq. (2) may be written in the divergence form in L^4 space as

$$\nabla \cdot B = C \tag{5}$$

The general curvilinear coordinate system may be enunciated by the following mapping:

$$\xi^i = \xi^i(x^1, x^2, x^3, x^4) \qquad i = 1,2,3$$

$$\xi^4 = x^4 \tag{6}$$

where $(x^1, x^2, x^3, x^4) = (x, y, z, t)$. In order to express Eq. (5) in curvilinear coordinates, use will be made of the following tensor formula for the divergence of a vector F:

$$\nabla \cdot F = \frac{1}{J} \frac{\partial}{\partial \xi^\alpha}(JF^\alpha) \tag{7}$$

where J is the Jacobian of the transformation from x^i to ξ^i.

By use of Eq. (7), Eq. (5), in a curvilinear coordinate system that includes time, may then be expressed as

$$\frac{\partial}{\partial \xi^\alpha}(JB^\alpha) = JC \tag{8}$$

The only difference in the treatment of the time and space variables is clear when it is observed that

$$\frac{\partial \xi^4}{\partial x^\beta} = \delta_{4\beta} \tag{9}$$

from Eq. (6). Thus,

$$B^4 = B_c^4 = A \tag{10}$$

$$B^i = B_c^\beta \frac{\partial \xi^i}{\partial x^\beta} = B_c^j \frac{\partial \xi^i}{\partial x^j} + B_c^4 \frac{\partial \xi^i}{\partial x^4}$$

$$= f_c^j \frac{\partial \xi^i}{\partial x^j} + A \frac{\partial \xi^i}{\partial t} = f^i + A\omega^i \tag{11}$$

§Except for x^i and ξ^j, the subscripts i, j, and k denote covariants, while the superscripts denote contravariant components in L^3. α and β have a similar interpretation in L^4. Repeated indices imply summation, while the subscript or superscript c denotes Cartesian components.

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where $\omega^i = \partial\xi^i/\partial t$. Substituting B^α in Eqs. (10) and (11) into Eq. (8), we obtain the expression in its usual form

$$\frac{\partial}{\partial t}(JA) + \frac{\partial}{\partial \xi^i}[J(f^i + A\omega^i)] = JC \tag{12}$$

or, in terms of Cartesian components,

$$\frac{\partial}{\partial t}(JA) + \frac{\partial}{\partial \xi^i}\left[J\left(f^j_c \frac{\partial \xi^i}{\partial x^j} + A\omega^i\right)\right] = JC \tag{13}$$

If A, C, and f^j are also vectors in L^3, by equating the components of E^c_i in Eq. (13), the scalar form may be obtained

$$\frac{\partial}{\partial t}(JA^k_c) + \frac{\partial}{\partial \xi^i}\left[J\left(f^{jk}_{cc} \frac{\partial \xi^i}{\partial x^j} + A^k_c\omega^i\right)\right] = JC^k_c \tag{14}$$

For a steady coordinate system ω^i vanishes.

Numerical Considerations

For numerical purposes, the terms $\partial\xi^\alpha/\partial x^\beta$ may be evaluated in terms of $\partial x^\alpha/\partial\xi^\beta$ from the following expression:

$$\frac{\partial \xi^\alpha}{\partial x^\beta} = \frac{R_{\alpha\beta}}{J} \tag{15}$$

Here $R_{\alpha\beta}$ are the algebraic complements of the Jacobian matrix $\partial x^\alpha/\partial\xi^\beta$.

On the other hand, the geometric conservation law[6] may also be obtained in a more direct and simpler fashion. It is noted that Eq. (2) is satisfied for any constants A, f, and zero C concurrently with Eq. (8), which gives

$$\frac{\partial}{\partial \xi^\alpha}\left(J\frac{\partial \xi^\alpha}{\partial x^\beta}\right) = \frac{\partial R_{\alpha\beta}}{\partial \xi^\alpha} = 0, \qquad \beta = 1,2,3,4 \tag{16}$$

In particular, $\beta = 4$ results in the geometric conservation law. Hence the necessary requirement of a conservative discrete operator that approximates Eq. (13) or (14) is such that Eq. (16) should be satisfied.

It is sufficient to mention here that the correct conservation forms of diverse fluid dynamic equations (e.g., the Navier-Stokes equations) can be obtained by selecting appropriate quantities for A, f, and C. For the case of two-dimensional equations, i, j, and k should denote the components in L^2 and α and β in L^3. Here, of course, the third added base vector denotes time t.

Concluding Remarks

A more direct method than has been reported previously to derive the conservative form of the equations of fluid dynamics in general nonsteady curvilinear coordinates has been presented. The geometric conservation law has also been derived as an aid to the development of compatible numerical algorithms.

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Finite Difference Solutions of the Euler Equations in the Vicinity of Sharp Edges

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Introduction

NUMEROUS attempts have been made to explain why finite difference solutions of the Euler equations can describe flows with large vortical structures around sharp-edged bodies. Some investigators[1,2] claim compressibility effects are the underlying mechanism for the generation of the necessary vorticity, others[3,4] suspect that artificial numerical damping is the causative agent.

The basic idea of the present approach is to study the influence of a singular sharp edge on the truncation error for a set of discretized Euler equations. First, the distribution of the truncation error of one finite difference approximation of the Euler equations near a sharp edge of a thin plate is analyzed. This leads to a determination of the size of the region of the neighborhood of such a singularity, where the leading terms of the truncation error are of the same order as the terms describing the changes in momentum and pressure. Finally, these results are verified in numerical experiments.

Consistency of a Discretization of the Euler Equations

The Euler equations in nondivergence formulation for an incompressible flow are written in nondimensional variables as

$$\frac{\mathrm{D}v}{\mathrm{D}t} = -\nabla p \tag{1}$$

with $v = (u'/v_\infty, v'/v_\infty, 0)$ and $p = p'/(\rho v_\infty^2)$. These equations are discretized explicitly with respect to time and with centered space difference quotients representing the spatial derivatives. They are solved by using a time-marching technique that will be described more in detail in the next section. Such a discretization of the Euler equations is unconditionally unstable unless numerical damping is added. For this reason, a generalized Lax method[4,5] is employed. The result-

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